

1] So Last time $\mathbb{N} \rightarrow \mathbb{R}$

Thm(Cantor): $\nexists$ a surj. $\mathbb{N} \rightarrow [0,1]$, ergo $\#\mathbb{R} > \aleph_0$.

Pf: Let $f: \mathbb{N}^+ \rightarrow [0,1]$. write

$$f(1) = . a_{11} a_{12} a_{13} \dots$$

$$f(2) = . a_{21} a_{22} a_{23} \dots$$

$$f(3) = . a_{31} a_{32} a_{33} \dots$$

$\vdots$

2] Define $X = .b_1 b_2 b_3 \dots$

$$b_i = \begin{cases} 0 & \text{if } a_{ii} \neq 0 \\ 1 & \text{if } a_{ii} = 0 \end{cases}$$

Note, $x \in [0, 1]$, but $x \neq f(i)$ for any i , $b < 1$

So, $x \in [0, 1] - f(\mathbb{N})$. $\square$

i^{th} digit of $x = b_i \neq a_{ii} = i^{\text{th}}$ digit of $f(i)$.

3] §1 Infinite vs. "at least $\aleph_0$ "

Prop: Let X be a set. $T \models A \in E$;

(1) X is not finite (i.e., $\nexists n \in \mathbb{N}$ for any $n \in \mathbb{N}$),

(2) $\#X \geq \aleph_0$.

PP: (1) $\Rightarrow$ (2). Want an $i: \mathbb{N} \rightarrow X$. Define elements $x_i \in X$ iteratively as follows.

4) $i=0$: $X \neq \emptyset$, Choose $x_0 \in X$.
 $i=1$: $x_1 \in X - \{x_0\} \neq \emptyset$
 $\vdots$
 $i=n$: ———
 $i=n+1$: Assume, $X - \{x_0, \dots, x_n\}$ is empty. Define $\{0, \dots, n\} \rightarrow X$
 which if $x_i \mapsto x_i$
 $x_{n+1} \in X - \{x_0, \dots, x_n\}$ is empty is a b.t.j. Contradiction. Choose
 Define $f: \mathbb{N} \rightarrow X$, $f(i) = x_i$ is a h.i.n.j.

5.1 (2) $\Rightarrow$ (1) There exists an inj. $N \rightarrow X$.
If X was finite $\exists$ a bij. $X \rightarrow \{0, \dots, n\}$.
So, $f: N \rightarrow \{0, \dots, n\}$ composition is an inj. The,
 $f: \{0, \dots, n+1\} \rightarrow \{0, \dots, n\}$ is still an inj. But
this is impossible by P.H.P. $\square$

6] §2 Subtracting finite sets from infinite

Thm: Suppose X is infinite and $S \subseteq X$ is finite
 $\#(X - S) = \#X$.

Pf: Case 1: $\#X = \aleph_0$.

Then, $\exists$ a bij. $f: X \rightarrow \mathbb{N}$. Note that as S is fin.
 it has $n+1$ elements for some n . WLOG $f(S) = \{0, \dots, n\}$.

[e.g. $S = \{x_0\}$, $f(x_0) = 117$.
 $\begin{array}{ccccccc} 0 & 1 & 2 & 3 & \dots & 117 & 118 & \dots \\ \downarrow & \downarrow & \downarrow & \downarrow & & \downarrow & \downarrow & \\ 117 & 1 & 2 & 3 & \dots & 0 & 118 & \dots \end{array}$

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But we then see

$$\#(X-S) = \#(N - \{0, \dots, n_0\})$$

$$\begin{array}{ccc} X & \supseteq & S \\ \downarrow & & \downarrow \\ N & \supseteq & \{0, \dots, n_0\} \end{array}$$

STS

$$\#(N - \{0, \dots, n_0\}) = \#N = \#X$$

$$\#(X-S) = \dots = \dots = \dots = \dots$$

But,

$$N \longrightarrow N - \{0, \dots, n_0\}, \quad n \longmapsto n + (n_0 + 1)$$

8] But if X is general, then by the previous

Prop $\exists$ an inj. $\mathbb{N} \rightarrow X$, call its image X_0 . WLOG.

$S \subseteq X_0$. Then

$$\begin{array}{lcl} \text{b1,} & X = (X - X_0) \sqcup X_0 & \\ & \downarrow \quad \downarrow \quad \downarrow & \\ & X - S = (X - X_0) \sqcup (X_0 - S) & \text{Previous case} \end{array}$$



§3 Cantor's (other) theorem

Def'n: If X and Y are sets,

$$Y^X := \{ f: X \rightarrow Y \}$$

If $Y = \{1, \dots, n\}$ then we shorten this to n^X .

e.g. $X = \{1, \dots, m\}$, $\# 2^X = \# \{ f: X \rightarrow \{1, 2\} \}$
 $= 2^m$

10- Prop: For any $X \ni$ a "natural"
bij

$$2^X \rightarrow P(X) = \{S \subseteq X\}$$

Pf: Let $S \subseteq X$. Its indicator function is

$$\mathbb{1}_S(x) = \begin{cases} 1 & \text{if } x \in S \\ 0 & \text{if } x \notin S \end{cases}$$

Then we define $P(X) \rightarrow 2^X$, $S \mapsto \mathbb{1}_S$. This is a
bij w/ inverse $f: X \rightarrow \{0, 1\}$ mapped $f^{-1}(1) \subseteq X$. $\square$

11. Thm (Cantor): $\# 2^X > \# X$

Pf: we show $\nexists$ a surj. $X \rightarrow P(X)$.

Let $f: X \rightarrow P(X)$ be any function. Let

$$S = \{x \in X : x \notin f(x)\}$$

If f was surj; $\exists s \in X$ s.t. $f(s) = S$.

12] But
• if $s \in S$ by def'n $s \neq f(s) = S$,
• if $s \notin S = f(s)$ so by def'n $s \in S$.

Either way we have a contradiction. So, S is
not in the image of f , ergo f is not sur; $\square$

$$\#P(\mathbb{N}) = \#\mathbb{R}$$